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BASIC PROPERTIES OF n- INNER PRODUCT SPACE

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ABSTRACT

In this paper we discuss certain fundamental properties of n-inner product space via an n – normed linear space.

Keywords: n-inner product**,** n-inner product space, n-normed product space.

Introduction:1.1

This paper is dealt with some properties of an n – inner product space $n \geq 2$. Also we establish the explicit forms of n – inner product space via an n – normed linear space. Some inter related results among n – normed linear space and n – inner product space also shown here.

Definition:1.2

Let "n" be a positive integer and X be a vector space of dimension $d \geq n$ (d may be infinite) over the field of real numbers R. A real valued function $\langle \cdot\, , \cdot \mid \cdot, \ldots, \cdot \rangle$ is defined on $X \times X \times \ldots \times X = X^{n+1}$ satisfying the following conditions

(I1) $\langle x_1, x_1 | x_2, \ldots, x_n \rangle \geq 0$ for any $x_1, x_2, \ldots, x_n \in X$ and

$\langle x_1, x_1 | x_2, \ldots, x_n \rangle = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent vectors.

(I2) $\langle x_1, x_1 | x_2, \ldots, x_n \rangle = \langle x_{i1}, x_{i1} | x_{i2}, \ldots, x_{in} \rangle$ for every permutation

$(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$

(I3) $\langle x, y | x_2, \ldots, x_n \rangle = \langle y, x | x_2, \ldots, x_n \rangle \; \forall \; x, y, x_2, \ldots, x_n \in X$

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(I4) $\langle \alpha x, y|x_2,\ldots,x_n\rangle = \alpha\langle x, y|x_2,\ldots,x_n\rangle\ \forall\ x_2,\ldots,x_n \in X, \forall\ \alpha \in R$

(I5) $\langle x+ y, z|x_2,\ldots,x_n\rangle = \langle x, z|x_2,\ldots,x_n\rangle + \langle y, z|x_2,\ldots,x_n\rangle$

$$\forall\ x, y, z, x_2,\ldots,x_n \in X$$

is called an n – inner product on X and the corresponding pair

$(X, \langle\cdot\,,\cdot\,|\,\cdot, \ldots, \cdot\rangle)$ called the n – inner product space.

Example:1.3

If $X = R^n$ then the following function

$$\langle x, y|x_2,\ldots,x_n\rangle = \begin{vmatrix} \langle x,y\rangle & \langle x,x_2\rangle & \ldots & \langle x,x_n\rangle \\ \langle x_2,y\rangle & \langle x_2,x_2\rangle & \ldots & \langle x_2,x_n\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \langle x_n,y\rangle & \langle x_n,x_n\rangle & \ldots & \langle x_n,x_n\rangle \end{vmatrix}$$

where $x, y, x_2,\ldots,x_n \in X$ defines an n – inner product, called the standard or (simple) n – inner product on X.

Some basic properties of n – inner product $(X, \langle\cdot\,,\cdot\,|\,\cdot, \ldots, \cdot\rangle)$ are as follows

(NIP1) $|\langle x, y|x_2,\ldots,x_n\rangle| = \sqrt{\langle x, x|x_2,\ldots,x_n\rangle}\sqrt{\langle y, y|x_2,\ldots,x_n\rangle}$

$\forall\ x, y, x_2,\ldots,x_n \in X$ and is known as an extension of the Cauchy – Schwartz inequality.

(NIP2) $\langle x, y|x_2,\ldots,x_n\rangle = 0\ \forall\ x, y,\ x_2,\ldots,x_n \in X$

(NIP3) $\langle x, y|\alpha x_2,\ldots,x_n\rangle = \alpha^2\langle x, y|x_2,\ldots,x_n\rangle\ \forall\ x, y, x_2,\ldots,x_n \in X$

and $\forall\ \alpha \in R$

(NIP4) $\langle x, y|z + w, x_2,\ldots,x_n\rangle = \langle x, y|z, x_2,\ldots,x_n\rangle + \langle x, y|w, x_2,\ldots,x_n\rangle$

$+\frac{1}{2}[\langle z, w|x + y, x_2,\ldots,x_n\rangle - \langle z, w|x - y, x_2,\ldots,x_n\rangle]$

$$\forall\ x, y, z,\ x_2,\ldots,x_n \in X$$

Definition:1.4

Let $(X, \langle\cdot\,,\cdot\,|\,\cdot, \ldots, \cdot\rangle)$ be an n – inner product space.

Let $(\|\cdot, \ldots, \cdot\|)$ be non negative real valued function $X \times X \times\ldots\times X = X^n :\to R$ satisfying the following conditions:

(i) $\|x_1, x_2,\ldots,x_n\| = 0$ if and only if $x_1, x_2,\ldots,x_n \in X$ are linearly dependent.

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(ii) $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation of $x_1, x_2, \ldots, x_n \in X$.
(iii) $\|x_1, x_2, \ldots, \alpha\, x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|$ for every $\alpha \in R$, $x_1, x_2, \ldots, x_n \in X$.
(iv) $\|x_1, x_2, \ldots, x_{n-1}, y + z\| \leq \|x_1, x_2, \ldots, x_{n-1}, y\| + \|x_1, x_2, \ldots, x_{n-1}, z\|$

for all $y, z, x_1, x_2, \ldots, x_{n-1} \in X$ then $\|\cdot, \ldots, \cdot\|$ is called an n – norm on X and the corresponding pair $(X, \|\cdot, \ldots, \cdot\|)$ is called n – normed linear space.

Example:1.5

The space $X = R^n$ equipped with the following n – norm.

$$\|x_1, x_2, \ldots, x_n\| = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ & \vdots & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ for each $i = 1, 2, \ldots, n$

Some basic properties of an n – normed space $(X, \|\cdot, \ldots, \cdot\|)$ are as follows:

(NN1) $\|x_1, x_2, \ldots, x_n\| \geq 0 \ \forall\, x_1, x_2, \ldots, x_n \in X$

(NN2) $\|x_1, x_2, \ldots, x_n + \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_{n-1} x_{n-1}\| = \|x_1, x_2, \ldots, x_n\|$

$$\forall\, x_1, x_2, \ldots, x_n \in X \ \forall\, \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in R$$

In any linear n – inner product space $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ we define an n – norm by $\|x_1, x_2, \ldots, x_n\| = \sqrt{\langle x_1, x_1 | x_2, \ldots, x_n \rangle}\ \forall x, y, x_2, \ldots, x_n \in X$

in which the following holds.

(NN3) $\|x + y, x_2, \ldots, x_n\|^2 + \|x - y, x_2, \ldots, x_n\|^2 = 2(\|x, x_2, \ldots, x_n\|^2 + \|y, x_2, \ldots, x_n\|^2)$

which is known as extension of parallelogram law.

(NN4) The Polarization identity:

$$\|x + y, x_2, \ldots, x_n\|^2 - \|x - y, x_2, \ldots, x_n\|^2 = 4\langle x, y | x_2, \ldots, x_n \rangle$$

By the Polarization identity and the property (I2) we observe that

$\langle x, y | x_2, \ldots, x_n \rangle = \langle x, y | x_{i2}, \ldots, x_{in} \rangle$, for every permutation $(i_2, \ldots, i_n)$ of $(2, 3, \ldots, n)$.

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Also $\langle x, y|x_2,\dots,x_n\rangle = 0$ when x or y is a linear combination of $x_2,\dots,x_n$ or when $x_2,\dots,x_n$ are linearly dependent.

(NN5) Just as in an inner product space, we have the Cauchy – Schwartz inequality.

$|\langle x, y|x_2,\dots,x_n\rangle| \le \|x, x_2,\dots,x_n\| \, \|y, x_2,\dots,x_n\|$

And the equality holds if and only if $x, y, x_1, x_2, \dots, x_n$ are linearly dependent.

Note:1.6

If $(X, \|\cdot,\dots,\cdot\|)$ is an n – normed linear space in which the condition $\|x+y, x_2, \dots, x_n\|^2 + \|x-y, x_2, \dots, x_n\|^2 = 2(\|x, x_2,\dots,x_n\|^2 + \|y, x_2,\dots,x_n\|^2)$ is satisfied for all $x, y, z, x_2, \dots, x_n \in X$ then n – inner product

$(\langle\cdot,\cdot|\cdot,\dots,\cdot\rangle)$ on X is defined by

$$\langle x, y|x_2,\dots,x_n\rangle = \frac{1}{4}(\|x+y, x_2,\dots,x_n\|^2 - \|x-y, x_2,\dots,x_n\|^2)$$

Some basic lemmas

Lemma:1.7

In n – inner product space, we have the following

(i) $\|x+y, y+z, x_3,\dots,x_n\| = \|x-z, y+z, x_3,\dots,x_n\| = \|x+y, x-z, x_3,\dots,x_n\|$

(ii) $\|x+y, y-z, x_3,\dots,x_n\| = \|x+z, y-z, x_3,\dots,x_n\| = \|x+y, x+z, x_3,\dots,x_n\|$

(iii) $\|x-y, y+z, x_3,\dots,x_n\| = \|x+z, y+z, x_3,\dots,x_n\| = \|x-y, x+z, x_3,\dots,x_n\|$

(iv) $\|x-y, y-z, x_3,\dots,x_n\| = \|x-z, y-z, x_3,\dots,x_n\| = \|x-y, x-z, x_3,\dots,x_n\|$

Proof:

(i) Consider $\|x+y, y+z, x_3,\dots,x_n\|$

$= \|(x+y) - (y+z), y+z, x_3,\dots,x_n\|$ by (NN2)

$= \|x-z, y+z, x_3,\dots,x_n\|$

Again, $\|x+y, y+z, x_3,\dots,x_n\|$

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$= \|x+y, (x+y) - (y+z), x_3, \ldots, x_n\|$ by (NN2)

$= \|x+y, x-z, x_3, \ldots, x_n\|$

(ii) Consider $\|x+y, y-z, x_3, \ldots, x_n\|$

$= \|(x+y) - (y-z), y-z, x_3, \ldots, x_n\|$ by (NN2)

$= \|x+z, y-z, x_3, \ldots, x_n\|$

Again, $\|x+y, y-z, x_3, \ldots, x_n\|$

$= \|x+y, (x+y) - (y-z), x_3, \ldots, x_n\|$ by (NN2)

$= \|x+y, x+z, x_3, \ldots, x_n\|$

(iii) Consider $\|x-y, y+z, x_3, \ldots, x_n\|$

$= \|(x-y) + (y+z), y+z, x_3, \ldots, x_n\|$ by (NN2)

$= \|x+z, y+z, x_3, \ldots, x_n\|$

Again, $\|x-y, y+z, x_3, \ldots, x_n\|$

$= \|x-y, (x-y) + (y+z), x_3, \ldots, x_n\|$ by (NN2)

$= \|x-y, x+z, x_3, \ldots, x_n\|$

(iv) Consider $\|x-y, y-z, x_3, \ldots, x_n\|$

$= \|(x-y) + (y-z), y-z, x_3, \ldots, x_n\|$ by (NN2)

$= \|x-z, y-z, x_3, \ldots, x_n\|$

Again, $\|x-y, y-z, x_3, \ldots, x_n\|$

$= \|x-y, (x-y) + (y-z), x_3, \ldots, x_n\|$ by (NN2)

$= \|x-y, x-z, x_3, \ldots, x_n\|$

Lemma:1.8

In any n – inner product space X, the followings hold:

(i) $\|x+y, y+z, x_3, \ldots, x_n\|^2 = \sum +2\langle x, y|z, x_3, \ldots, x_n\rangle$
$-2\langle x, z|y, x_3, \ldots, x_n\rangle + 2\langle y, z|x, x_3, \ldots, x_n\rangle$

(ii) $\|x+y, y-z, x_3, \ldots, x_n\|^2 = \sum +2\langle x, y|z, x_3, \ldots, x_n\rangle$
$+2\langle x, z|y, x_3, \ldots, x_n\rangle - 2\langle y, z|x, x_3, \ldots, x_n\rangle$

(iii) $\|x-y, y+z, x_3, \ldots, x_n\| = \sum -2\langle x, y|z, x_3, \ldots, x_n\rangle$
$-2\langle x, z|y, x_3, \ldots, x_n\rangle + 2\langle y, z|x, x_3, \ldots, x_n\rangle$

(iv) $\|x-y, y-z, x_3, \ldots, x_n\| = \sum -2\langle x, y|z, x_3, \ldots, x_n\rangle$
$+2\langle x, z|y, x_3, \ldots, x_n\rangle - 2\langle y, z|x, x_3, \ldots, x_n\rangle$

Where $\sum = \|x, y, x_3, \ldots, x_n\|^2 + \|x, z, x_3, \ldots, x_n\|^2 + \|y, z, x_3, \ldots, x_n\|^2$

Proof:

(i) $\|x+y, y+z, x_3, \ldots, x_n\|^2 = \langle x+y, x+y|y+z, x_3, \ldots, x_n\rangle$

$= \langle x, x+y|y+z, x_3, \ldots, x_n\rangle + \langle y, x+y|y+z, x_3, \ldots, x_n\rangle$

$= \langle x, x|y+z, x_3, \ldots, x_n\rangle + \langle x, y|y+z, x_3, \ldots, x_n\rangle$

$+\langle y, x|y+z, x_3, \ldots, x_n\rangle + \langle y, y|y+z, x_3, \ldots, x_n\rangle$

$= \langle y+z, y+z|x, x_3, \ldots, x_n\rangle + \langle y+z, y+z|y, x_3, \ldots, x_n\rangle$

$+2\langle x, y|y+z, x, x_3, \ldots, x_n\rangle$

$= \langle y, y+z|x, x_3, \ldots, x_n\rangle + \langle z, y+z|x, x_3, \ldots, x_n\rangle$

$+\langle y, y+z|y, x_3, \ldots, x_n\rangle + \langle z, y+z|y, x_3, \ldots, x_n\rangle$

$+2\langle x, y|y+z, x_3, \ldots, x_n\rangle$

$= \langle y, y|x, x_3, \ldots, x_n\rangle + \langle y, z|x, x_3, \ldots, x_n\rangle + \langle z, y|x, x_3, \ldots, x_n\rangle$

$+\langle z, z|x, x_3, \ldots, x_n\rangle + \langle y, y|y, x_3, \ldots, x_n\rangle + \langle y, z|y, x_3, \ldots, x_n\rangle$

$+\langle z, y|y, x_3, \ldots, x_n\rangle + \langle z, z|y, x_3, \ldots, x_n\rangle + 2\langle x, y|y+z, x_3, \ldots, x_n\rangle$

$= \|y, x, x_3, \ldots, x_n\|^2 + \|y, z, x_3, \ldots, x_n\|^2 + \|z, x, x_3, \ldots, x_n\|^2$

$+2\langle y, z|x, x_3, \ldots, x_n\rangle + 2\langle x, y|y+z, x_3, \ldots, x_n\rangle$

Now, $\langle x, y|y+z, x_3, \ldots, x_n\rangle = \langle x, y|y, x_3, \ldots, x_n\rangle + \langle x, y|z, x_3, \ldots, x_n\rangle$

$+\frac{1}{2}[\langle y, z|x+y, x_3, \ldots, x_n\rangle - \langle y, z|x-y, x_3, \ldots, x_n\rangle]$

$= \langle x, y|z, x_3, \ldots, x_n\rangle + \frac{1}{2}[\langle y, z|x+y, x_3, \ldots, x_n\rangle - \langle y, z|x-y, x_3, \ldots, x_n\rangle]$

Also, $\langle y, z|x+y, x_3, \ldots, x_n\rangle = \langle x+y-x, z|x+y, x_3, \ldots, x_n\rangle$

$= \langle x+y, z|x+y, x_3, \ldots, x_n\rangle - \langle x, z|x+y, x_3, \ldots, x_n\rangle$

$= -\langle x, z|x+y, x_3, \ldots, x_n\rangle$

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$$\langle y,z|x-y,x_3,\dots,x_n\rangle = -\langle x-y-x,z|x-y,x_3,\dots,x_n\rangle$$
$$= -\langle x-y,z|x-y,x_3,\dots,x_n\rangle + \langle x,z|x-y,x_3,\dots,x_n\rangle$$
$$= \langle x,z|x-y,x_3,\dots,x_n\rangle$$

Now, $\langle x,y|y+z,x_3,\dots,x_n\rangle = \langle x,y|z,x_3,\dots,x_n\rangle +$

$$-\tfrac{1}{2}[\langle x,z|x+y,x_3,\dots,x_n\rangle - \langle x,z|x-y,x_3,\dots,x_n\rangle]$$
$$= \langle x,y|z,x_3,\dots,x_n\rangle - \tfrac{1}{2}[\langle x,z|x,x_3,\dots,x_n\rangle + \langle x,z|y,x_3,\dots,x_n\rangle$$
$$+\tfrac{1}{2}(\langle x,y|x+z,x_3,\dots,x_n\rangle - \langle x,y|x-z,x_3,\dots,x_n\rangle)]$$
$$+\tfrac{1}{2}[\langle x,z|x,x_3,\dots,x_n\rangle + \langle x,z|-y,x_3,\dots,x_n\rangle$$
$$+\tfrac{1}{2}(\langle x,-y,z|x+z,x_3,\dots,x_n\rangle - \langle x,-y|x-z,x_3,\dots,x_n\rangle)]$$
$$= \langle x,y|z,x_3,\dots,x_n\rangle - \langle x,z|y,x_3,\dots,x_n\rangle$$

Therefore, we have

$$\|x+y,y+z,x_3,\dots,x_n\|^2 = \sum +2\langle x,y|z,x_3,\dots,x_n\rangle - 2\langle x,z|y,x_3,\dots,x_n\rangle$$
$$+2\langle y,z|x,x_3,\dots,x_n\rangle$$

Now from Lemma 2.7 we have

$$4\sum = \|x+y,y+z,x_3,\dots,x_n\|^2 + \|x+y,y-z,x_3,\dots,x_n\|^2$$
$$+\|x-y,y+z,x_3,\dots,x_n\|^2 + \|x-y,y-z,x_3,\dots,x_n\|^2 \quad \dots \text{(I)}$$

$$8\langle x,y|z,x_3,\dots,x_n\rangle = [\|x+y,y+z,x_3,\dots,x_n\|^2$$
$$+ \|x+y,y-z,x_3,\dots,x_n\|^2]$$
$$-[\|x-y,y+z,x_3,\dots,x_n\|^2 + \|x-y,y-z,x_3,\dots,x_n\|^2] \quad \dots \text{(II)}$$

Theorem: 1.9

An n – normed linear space X is an n – inner product space if and only if (I) is true and n – inner product is given by (II).

Proof:

Suppose X is an n – inner product space. Then by lemma 2.7 (I) follows.

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Assume (I) is true in an n – normed linear space X. using (I) we have

(A): $4[\|z+y,x,x_3,\ldots,x_n\|^2 + \|x,z-y,x_3,\ldots,x_n\|^2 + \|z+y,z-y,x_3,\ldots,x_n\|^2]$

$$= \|x+y+z,2z,x_3,\ldots,x_n\|^2 + \|x+y+z,2y,x_3,\ldots,x_n\|^2 + \|x-y-z,2z,x_3,\ldots,x_n\|^2 + \|x-y-z,2y,x_3,\ldots,x_n\|^2$$

$$= 4[\|x+y+z,z,x_3,\ldots,x_n\|^2 + \|x+y+z,y,x_3,\ldots,x_n\|^2 + \|x-y-z,z,x_3,\ldots,x_n\|^2 + \|x-y-z,y,x_3,\ldots,x_n\|^2]$$

$$= 4[\|z,x+y,x_3,\ldots,x_n\|^2 + \|y,x+z,x_3,\ldots,x_n\|^2 + \|z,x-y,x_3,\ldots,x_n\|^2 + \|y,x-z,x_3,\ldots,x_n\|^2]$$

(B): $4[\|z+x,y,x_3,\ldots,x_n\|^2 + \|z-x,y,x_3,\ldots,x_n\|^2 + \|z+x,z-x,x_3,\ldots,x_n\|^2]$

$$= \|x+y+z,2z,x_3,\ldots,x_n\|^2 + \|x+y+z,2x,x_3,\ldots,x_n\|^2 + \|z+x-y,2z,x_3,\ldots,x_n\|^2 + \|z+x-y,2x,x_3,\ldots,x_n\|^2$$

$$= 4[\|x+y+z,z,x_3,\ldots,x_n\|^2 + \|x+y+z,x,x_3,\ldots,x_n\|^2 + \|z+x-y,z,x_3,\ldots,x_n\|^2 + \|z+x-y,x,x_3,\ldots,x_n\|^2]$$

$$= 4[\|z,y+x,x_3,\ldots,x_n\|^2 + \|x,y+z,x_3,\ldots,x_n\|^2 + \|z,y-x,x_3,\ldots,x_n\|^2 + \|x,y-z,x_3,\ldots,x_n\|^2]$$

Adding (A) and (B) we have

$$\|x+y,z,x_3,\ldots,x_n\|^2 + \|x-y,z,x_3,\ldots,x_n\|^2 = 2[\|x,z,x_3,\ldots,x_n\|^2 + \|y,z,x_3,\ldots,x_n\|^2]$$

Therefore we have an n – inner product space with

$$4\langle x,y/z,x_3,\ldots,x_n\rangle = \frac{1}{4}[\|x+y,z,x_3,\ldots,x_n\|^2 - \|x-y,z,x_3,\ldots,x_n\|^2]$$

Once again using (I) we have

(C): $4[\|x+y,y+z,x_3,\ldots,x_n\|^2 + \|x+y,y-z,x_3,\ldots,x_n\|^2 +$

$$\|y+z, y-z, x_3,\ldots,x_n\|^2]$$

$$= \|x+2y+z, 2y, x_3,\ldots,x_n\|^2 + \|x+2y+z, 2z, x_3,\ldots,x_n\|^2$$

$$+\|x-z, 2y, x_3,\ldots,x_n\|^2 + \|x-z, 2z, x_3,\ldots,x_n\|^2$$

$$= 4[\|x+2y+z, y, x_3,\ldots,x_n\|^2 + \|x+2y+z, z, x_3,\ldots,x_n\|^2$$

$$+\|x-z, y, x_3,\ldots,x_n\|^2 + \|x-z, z, x_3,\ldots,x_n\|^2]$$

$$= 4[\|x+z, y, x_3,\ldots,x_n\|^2 + \|x+2y, z, x_3,\ldots,x_n\|^2$$

$$+\|x-z, y, x_3,\ldots,x_n\|^2 + \|x, z, x_3,\ldots,x_n\|^2]$$

(D): $4[\|x-y, y+z, x_3,\ldots,x_n\|^2 + \|x-y, y-z, x_3,\ldots,x_n\|^2 -$

$$\|y+z, y-z, x_3,\ldots,x_n\|^2]$$

$$= \|x+z, 2y, x_3,\ldots,x_n\|^2 + \|x+z, 2z, x_3,\ldots,x_n\|^2$$

$$+\|x-2y-z, 2y, x_3,\ldots,x_n\| + \|x-2y-z, 2z, x_3,\ldots,x_n\|^2$$

$$= 4[\|x+z, y, x_3,\ldots,x_n\|^2 + \|x+z, z, x_3,\ldots,x_n\|^2$$

$$+\|x-2y-z, y, x_3,\ldots,x_n\| + \|x-2y-z, z, x_3,\ldots,x_n\|^2]$$

$$= 4[\|x+z, y, x_3,\ldots,x_n\|^2 + \|x, z, x_3,\ldots,x_n\|^2$$

$$+\|x-z, y, x_3,\ldots,x_n\|^2 + \|x-2y, z, x_3,\ldots,x_n\|^2]$$

Subtracting (D) from (C) and using (II) we get,

$$\langle x, y|z, x_3,\ldots,x_n\rangle = \frac{1}{8}[\|x+2y, z, x_3,\ldots,x_n\|^2 - \|x-2y, z, x_3,\ldots,x_n\|^2]$$

$$= \frac{1}{2}\langle x, 2y|z, x_3,\ldots,x_n\rangle$$

$$= \langle x, y|z, x_3,\ldots,x_n\rangle$$

This completes the proof.

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